

ON GROUPS WITH CUBIC POLYNOMIAL CONDITIONS

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ABSTRACT. Let F_d be the free group of rank d , freely generated by $\{y_1, \dots, y_d\}$, and let $\mathbb{D}F_d$ be the group ring over an integral domain \mathbb{D} . Given a subset E_d of F_d containing the generating set, assign to each s in E_d a monic polynomial $p_s(x) = x^n + c_{s,n-1}x^{n-1} + \dots + c_{s,1}x + c_{s,0} \in \mathbb{D}[x]$ and define the quotient ring

$$A(d, n, E_d) = \frac{\mathbb{D}F_d}{\langle p_s(s) \mid s \in E_d \rangle_{ideal}}.$$

When $p_s(s)$ is cubic for all s , we construct a finite set E_d such that $A(d, n, E_d)$ has finite rank over an extension of \mathbb{D} by inverses of some of the coefficients of the polynomials. When the polynomials are all equal to $(x-1)^3$ and $\mathbb{D} = \mathbb{Z}[\frac{1}{6}]$, we construct a finite subset P_d of F_d such that the quotient ring $A(d, 3, P_d)$ has finite \mathbb{D} -rank and its augmentation ideal is nilpotent. The set P_2 is $\{y_1, y_2, y_1y_2, y_1^{-1}y_2, y_1^2y_2, y_1y_2^2, [y_1, y_2]\}$ and we prove that $(x-1)^3 = 0$ is satisfied by all elements in the image of F_2 in $A(2, 3, P_d)$.

1. INTRODUCTION

The impact of finite order conditions on a group has guided major developments in group theory and so have similar finiteness questions in the theory of algebras [1]. The purpose of this paper is to examine finitely generated groups where a finite number of its elements satisfy polynomial equations in one variable in degrees 2 and 3.

The precise setting is as follows: we start with a free group F_d of rank d , freely generated by $\{y_1, \dots, y_d\}$, with the group ring $\mathbb{D}F_d$ over an integral domain \mathbb{D} and with a set E_d of elements of F_d containing the generating set. We then assign to each s in E_d a monic polynomial $p_s(x) = x^n + c_{s,n-1}x^{n-1} + \dots + c_{s,1}x + c_{s,0} \in \mathbb{D}[x]$ and define the quotient ring

$$A(d, n, E_d) = \frac{\mathbb{D}F_d}{\langle p_s(s) \mid s \in E_d \rangle_{ideal}}.$$

If $c_{s,0} = 0$ then s satisfies a polynomial of degree less than n . If $c_{s,0}$ is not zero then we may replace \mathbb{D} by its extension by the inverse of $c_{s,0}$.

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Let a_1, \dots, a_d be the respective images of y_1, \dots, y_d in $\mathbf{A}_d = A(d, n, E_d)$, let $G_d = \langle a_1, \dots, a_d \rangle$ and let $\mathbf{B}_d = \omega(\mathbf{A}_d)$ be the image of the augmentation ideal of $\mathbb{D}F_d$; recall that \mathbf{B}_d is generated by $u(g) = g - 1$ for all $g \in G$.

First we prove a finiteness rank condition for $n = 3$.

Theorem 1. *Define the following subsets of F_d*

$$E_1 = \{y_1\}, \quad M_1 = \{e, y_1^{\pm 1}\}$$

and inductively for $1 \leq s \leq d-1$,

$$\begin{aligned} E_{s+1} &= E_s \cup M_s y_{s+1}^{\pm 1}, \\ M_{s+1} &= M_s \cup M_s y_{s+1}^{\pm 1} M_s \\ &\quad \cup M_s y_{s+1}^{-1} (M_s \setminus \{e\}) y_{s+1} M_s. \end{aligned}$$

Suppose that each $s \in E_d$ satisfies some cubic polynomial $p_s(x) = x^3 + c_{s,2}x^2 + c_{s,1}x + c_{s,0} \in \mathbb{D}[x]$. Then (i) $A(d, n, E_d)$ is the linear span of the images of M_d over an extension of \mathbb{D} by inverses of some of the coefficients; (ii) the set E_2 is $\{y_1, y_2, y_1 y_2, y_1 y_2^{-1}\}$ and for general d , the elements of E_d are primitive in the generators $\{y_1, \dots, y_d\}$; (iii) $|M_2| \leq 39$ and $\log_3 |M_d| \leq 4.3^{d-2}$ for all d .

Since the d -generated Burnside group $B(d, 3)$ of exponent 3 satisfies the unipotent condition $(x-1)^3 = 0$ in its group algebra over $GF(3)$, the mentioned logarithmic upper bound is at least $\log_3 |B(d, 3)| = d + \binom{d}{2} + \binom{d}{3}$ ([3], page 88).

The rest of the paper is a study of the quotient rings $A(d, n, E_d)$ for unipotent polynomials $p_s(x) = (x-1)^n$ where $n = 2, 3$.

In case $n = 2$ we prove:

Theorem 2. *Define the quotient ring $\mathbf{A}_d = A(d, 2, S_d) = \frac{\mathbb{Z}F_d}{\langle (x-1)^2 | x \in S_d \rangle}$ where*

$$S_d = \{y_i \mid (1 \leq i \leq d), y_i y_j \mid (1 \leq i < j \leq d)\}.$$

Then: (i) the map

$$\begin{aligned} \varphi &: a_1 \rightarrow a_{1,d} = \begin{pmatrix} I_{2^{d-1}} & 0 \\ I_{2^{d-1}} & I_{2^{d-1}} \end{pmatrix}, \\ a_i &\rightarrow a_{i,d} = \begin{pmatrix} a_{i-1,d-1} & 0 \\ a_{i-1,d-1} & (a_{i-1,d-1})^{-1} \end{pmatrix} \quad (2 \leq i \leq d) \end{aligned}$$

extends to a monomorphism $\varphi : \mathbf{A}_d \rightarrow M(2^d, \mathbb{Z})$ and \mathbf{A}_d is a free \mathbb{Z} -module of rank 2^d ; (ii) \mathbf{B}_d is nilpotent of degree $d+1$, $(\mathbf{B}_d)^d = \mathbb{Z}u(a_1) \dots u(a_d)$; (iii) G_d is a free d -generated nilpotent group of class 2.

In case $n = 3$, first we prove:

Theorem 3. *Let $\mathbb{D} = \mathbb{Z}[\frac{1}{6}]$. Consider the quotient ring $\mathbf{A} = A(2, 3, P) = \frac{\mathbb{D}F_2}{\langle (x-1)^3 | x \in P \rangle}$ where*

$$P = \{y_1, y_2, y_1 y_2, y_1^{-1} y_2, y_1^2 y_2, y_1 y_2^2, [y_1, y_2]\},$$

Then

- (i) $\text{rank } \mathbf{A} = 18$,
- (ii) the ideal \mathbf{B} is nilpotent of degree 6,
- (iii) G is a free 2-generated nilpotent group of degree 4,
- (iv) $(x-1)^3 = 0$ is satisfied by all elements of G .

The development of relations in the proof of Theorem 3 is an abridged form of that presented in [4] where we traced the effect of gradually introducing relations to $\mathbb{Z}F_2$ and also detected the appearance of torsion conditions in the ring. There, we calculated relations by hand and confirmed them by using the computer program GAP [5]. In particular, we used the package GBNP [7] along with some special substitution routines [6]. If the analysis were carried further, it would provide a complete description of $\frac{\mathbb{Z}F_2}{\langle (x-1)^3 | x \in P \rangle}$. In the present paper we take the shorter route by analyzing directly the ring \mathbf{A} . In certain places, when the computational procedure becomes clear, we refer to [4] for more details. Item (iv) of the theorem follows from a determination of the set of solutions of $X^3 = 0$ in \mathbf{B} , as an algebraic affine variety over \mathbb{D} .

It follows from a deep theorem of Zelmanov on Lie algebras satisfying an n -Engel identity over a field k of characteristic zero, that if all the elements of a finitely generated group H satisfy $(x-1)^n = 0$ then H is nilpotent ([2], page 71-72). This result is also true for fields k of finite characteristic p which is large enough. The lower bound for p provided by the proof is super-exponential in n ; it was conjectured by Zelmanov that $p \geq 2n$ is sufficient.

We conclude the paper with:

Theorem 4. *Let $\mathbb{D} = \mathbb{Z}[\frac{1}{6}]$. There exists a finite subset P_d of F_d , extending P , such that the quotient ring $\mathbf{A}_d = A(d, 3, P_d) = \frac{\mathbb{D}F_d}{\langle (x-1)^3 | x \in P_d \rangle}$ has finite \mathbb{D} -rank and the augmentation ideal \mathbf{B}_d is nilpotent.*

We hope that the approach we have taken in this paper may be applied to Burnside groups of exponent 5 satisfying $(x-1)^4 = 0$ in characteristic 5. For recent works on this topic, see [8], [9].

2. GENERAL CUBIC CONDITIONS (PROOF OF THEOREM 1)

(i) We treat first the case $d = 2$. Write $a_1 = a, a_2 = b$. We may assume each $x \in \{a, b, ab, a^{-1}b\}$ satisfies

$$x^3 = \delta_x x^2 + \gamma_x x + \varepsilon_x$$

for some $\gamma_x, \delta_x, \varepsilon_x$ in \mathbb{D} and ε_x invertible in \mathbb{D} .

Multiplying the above by x^{-1} produces

$$x^2 = \varepsilon_x x^{-1} + \delta_x x + \gamma_x,$$

and by x^{-3} produces

$$x^{-3} = -\varepsilon_x^{-1} (\gamma_x x^{-2} + \delta_x x^{-1} - 1);$$

that is, x^{-1} satisfies a cubic polynomial over \mathbb{D} . More generally, x^k is a linear combination of $\{1, x, x^{-1}\}$ for all integers k . Therefore, the ring generated by G is a \mathbb{D} -linear combination of monomials in a^i, b^i ($i = 0, \pm 1$). Conjugation of

$$(ab)^3 = \delta_{ab} (ab)^2 + \gamma_{ab} (ab) + \varepsilon_{ab}$$

by a^{-1} produces

$$(ba)^3 = \delta_{ab} (ba)^2 + \gamma_{ab} (ba) + \varepsilon_{ab}$$

and inversion produces

$$(a^{-1}b^{-1})^3 = -\varepsilon_{ab}^{-1} (\gamma_{ab} (a^{-1}b^{-1})^2 + \delta_{ab} (a^{-1}b^{-1}) - 1).$$

Similarly, the formula for $(a^{-1}b)^3$ produces

$$(ab^{-1})^3 = -\varepsilon_{a^{-1}b} \left(\gamma_{a^{-1}b} (ab^{-1})^2 + \delta_{a^{-1}b} (ab^{-1}) - 1 \right).$$

That is, every $a^i b^j$ ($i, j = \pm 1$) satisfies a cubic polynomial. On multiplying the formula

$$(ab)^3 = \delta_{ab} (ab)^2 + \gamma_{ab} ab + \varepsilon_{ab}$$

by a^{-1} on the left and by $(ab)^{-1}$ on the right, we obtain

$$bab = \delta_{ab} b + \gamma_{ab} a^{-1} + \varepsilon_{ab} a^{-1} b^{-1} a^{-1}.$$

On substituting in the expressions for a^{-1}, b^{-1} we obtain a formula for bab as a linear combination of $a^i b^j a^k$ ($i, j, k = 0, \pm 1$).

Similarly, from the formulas for $(a^\varepsilon b^\delta)^3$ ($\varepsilon, \delta = -1, 1$) we obtain $b^m a^l b^m$ ($l, m = 1, 2$) as a linear combination of $a^i b^j a^k$ ($i, j, k = 0, \pm 1$). Therefore $b^m a^l b^{m+1}$ is a sum of monomials $a^i b^j a^k b$ ($i, j, k = 0, \pm 1$), from which set we can exclude $a^i b a^k b$ ($k = \pm 1$).

We assert that a monomial of type $w = \alpha_1 b^j \alpha_2 b^k \alpha_3 b^l \alpha_4$ where $\alpha_i = a^{\pm 1}$ ($1 \leq i \leq 4$) and having b -syllable length $|j| + |k| + |l| \leq 3$ is a sum of monomials of type $\alpha_1 b^r \alpha_2 b^s \alpha_3$ with $r, s = \pm 1$ and $(r, s) \neq (1, -1)$. From our formulas, we may assume that $j \neq k \neq l$. Typically, $w = \alpha_1 b a b^{-1} \alpha_2 b \alpha_3$. Then,

$$\begin{aligned} \alpha_1 b a b^{-1} \alpha_2 b \alpha_3 &= \alpha_1 (b a b^{-1}) \alpha_2 b \alpha_3 = \alpha_1 \left(\sum a^i b^j a^k b \right) \alpha_2 b \alpha_3 = \\ &= \sum \alpha_1 a^i . b^j a^k (b \alpha_2 b) \alpha_3 \\ &= \sum \alpha_1' b^j a^k \left(\sum \alpha_2' b^{j'} \alpha_2'' \right) \alpha_3 = \sum \alpha_1' b^j a^{k'} b^{j''} \alpha_3. \end{aligned}$$

For the general case, we argue by induction on d . Here, we need to consider only monomials of the form $w_1 a_1^{l_1} w_2 a_2^{l_2} \dots w_d a_d^{l_d}$ where $l_i = \pm 1$ and $w_i \in M_{d-1}$. The proof proceeds as before: $a_d^m w a_d^m$ ($m = \pm 1$) is a sum of monomials $w^i a_d^m w^k$ ($i, k = 0, 1, 2$) and $a_d^m w a_d^{m+1}$ is a sum of monomials $w^i a_d^m w^k a_d$ from which we can exclude $w^i a_d w^k a_d$.

(ii) The assertion here is clear.

(iii) That $|M_2| \leq 39$ is also clear. For general d , the cardinality of $|M_d|$ is estimated as follows

$$\begin{aligned} |M_{s+1}| &\leq |M_s| + |M_s|^2 + |M_s|^3 \\ &\leq \frac{|M_s|}{|M_s| - 1} (|M_s|^3 - 1) \end{aligned}$$

and by induction on d ,

$$\log_3 |M_d| \leq 4 \cdot 3^{d-2}.$$

3. QUADRATIC UNIPOTENT CONDITIONS (PROOF OF THEOREM 2)

(0) Let H be a subgroup of a ring S with unity and let $x \in H$ satisfy $u(x)^2 = 0$. Then

$$u(x^k) = k u(x) \text{ for all integers } k.$$

Suppose $x, y \in H$ satisfy

$$u(x)^2 = u(y)^2 = u(xy)^2 = 0.$$

Then

$$\begin{aligned}
u\left((xy)^{-1}\right) &= -u(yx) = -(u(y) + u(x) + u(y)u(x)), \\
u\left(x^{-1}y^{-1}\right) &= u\left(x^{-1}\right) + u\left(y^{-1}\right) + u\left(x^{-1}\right)u\left(y^{-1}\right) \\
&= -u(x) - u(y) + u(x)u(y), \\
u(y)u(x) &= -u(x)u(y);
\end{aligned}$$

$$\begin{aligned}
u([x, y]) &= u\left(x^{-1}y^{-1}xy\right) \\
&= u\left(x^{-1}\right) + u\left(y^{-1}\right) + u(x) + u(y) \\
&\quad + u\left(x^{-1}\right)u\left(y^{-1}\right) + u\left(x^{-1}\right)u(x) + u\left(x^{-1}\right)u(y) \\
&\quad + u\left(y^{-1}\right)u(x) + u\left(y^{-1}\right)u(y) \\
&\quad + u(x)u(y) \\
&= u(x)u(y) - u(y)u(x) = 2u(x)u(y);
\end{aligned}$$

$$u([x, y])^2 = 0.$$

(i) In A_d ,

$$a_i^2 = 2a_i - 1,$$

for all i and

$$\begin{aligned}
u(a_j)u(a_i) &= -u(a_i)u(a_j), \\
a_ja_i &= -a_ia_j + 2a_i + 2a_j - 2
\end{aligned}$$

for all i, j . Therefore, \mathbf{A}_d is a \mathbb{Z} -linear combination of 1 together with the monomials $a_{i_1}a_{i_2}\dots a_{i_s}$ for $1 \leq i_1 < i_2 < \dots < i_s \leq d$. We note that \mathbf{A}_d maps onto the group ring $\mathbb{Z}_{\overline{FV}}$ and therefore $\text{rank}_{\mathbb{Z}}(\mathbf{A}_d) = 2^d$.

(ii) Given the map

$$\begin{aligned}
\varphi \quad : \quad a_1 &\rightarrow a_{1,d} = \begin{pmatrix} I_{2^{d-1}} & 0 \\ I_{2^{d-1}} & I_{2^{d-1}} \end{pmatrix}, \\
a_i &\rightarrow a_{i,d} = \begin{pmatrix} a_{i-1,d-1} & 0 \\ a_{i-1,d-1} & (a_{i-1,d-1})^{-1} \end{pmatrix} \quad (2 \leq i \leq d),
\end{aligned}$$

we calculate

$$\begin{aligned}
(a_{i,d})^{-1} &= \begin{pmatrix} (a_{i-1,d-1})^{-1} & 0 \\ -a_{i-1,d-1} & a_{i-1,d-1} \end{pmatrix}, \\
a_{i,d} + (a_{i,d})^{-1} &= \begin{pmatrix} a_{i-1,d-1} + (a_{i-1,d-1})^{-1} & 0 \\ 0 & a_{i-1,d-1} + (a_{i-1,d-1})^{-1} \end{pmatrix} \\
&= 2I_{2^d}.
\end{aligned}$$

By induction on d , we have

$$(a_{i,d} - I_{2^d})^2 = \begin{pmatrix} (a_{i-1,d-1} - I_{2^{d-1}})^2 & 0 \\ a_{i-1,d-1} \left(a_{i-1,d-1} + (a_{i-1,d-1})^{-1} - 2I_{2^{d-1}} \right) & \left((a_{i-1,d-1})^{-1} - I_{2^{d-1}} \right)^2 \end{pmatrix} = 0.$$

Furthermore,

$$\begin{aligned}
a_{i,d}a_{j,d} &= \begin{pmatrix} a_{i-1,d-1} & 0 \\ a_{i-1,d-1} & (a_{i-1,d-1})^{-1} \end{pmatrix} \begin{pmatrix} a_{j-1,d-1} & 0 \\ a_{j-1,d-1} & (a_{j-1,d-1})^{-1} \end{pmatrix} \\
&= \begin{pmatrix} a_{i-1,d-1}a_{j-1,d-1} & 0 \\ \left(a_{i-1,d-1} + (a_{i-1,d-1})^{-1}\right)a_{j-1,d-1} & (a_{i-1,d-1})^{-1}(a_{j-1,d-1})^{-1} \end{pmatrix} \\
&= \begin{pmatrix} a_{i-1,d-1}a_{j-1,d-1} & 0 \\ 2a_{j-1,d-1} & (a_{i-1,d-1})^{-1}(a_{j-1,d-1})^{-1} \end{pmatrix} \\
a_{i,d}a_{j,d} - I_{2^d} &= \begin{pmatrix} (a_{i-1,d-1}a_{j-1,d-1} - I_{2^{d-1}}) & 0 \\ 2a_{j-1,d-1} & -a_{j-1,d-1}a_{i-1,d-1} + I_{2^{d-1}} \end{pmatrix}
\end{aligned}$$

Therefore,

$$\begin{aligned}
(a_{i,d}a_{j,d} - I_{2^d})^2 &= \begin{pmatrix} (a_{i-1,d-1}a_{j-1,d-1} - I_{2^{d-1}})^2 & 0 \\ \zeta & (-a_{j-1,d-1}a_{i-1,d-1} + I_{2^{d-1}})^2 \end{pmatrix}, \\
\zeta &= 2a_{j-1,d-1}(a_{i-1,d-1}a_{j-1,d-1} - I_{2^{d-1}}) + (-a_{j-1,d-1}a_{i-1,d-1} + I_{2^{d-1}})2a_{j-1,d-1} \\
&= 0, \\
(a_{i,d}a_{j,d} - I_{2^d})^2 &= 0.
\end{aligned}$$

Thus φ extends to a ring homomorphism $\mathbf{A}_d \rightarrow M(2^d, \mathbb{Z})$.

Observe that

$$\mathbf{A}_d = \mathbb{Z}\langle a_1, \dots, a_{d-1} \rangle + \mathbb{Z}\langle a_1, \dots, a_{d-1} \rangle a_d$$

which will be shown to be a direct sum. We do this first in the representation $(\mathbf{A}_d)^\varphi$.

Lemma 1. *Let $T_0 = \mathbb{Z}$ and for $s \geq 1$, $T_s = (\mathbf{A}_s)^\varphi$ and $U_{s+1} = T_s \otimes I_{2^{s+1}}$. Then, $U_{s+1} \cong T_s$ and*

$$T_d = U_d \oplus U_d \alpha_{d,d} = U_d \oplus U_d (\alpha_{d,d})^{-1}.$$

Proof. Let $d = 1$. Then, $T_1 = \mathbb{Z} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{Z} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. In general we have

$$T_d = U_d + U_d a_{d,d}$$

and since $(\alpha_{d,d})^{-1} = 2 - \alpha_{d,d}$,

$$T_d = U_d + U_d (a_{d,d})^{-1}.$$

Let $X, Y \in U_d$ and $\lambda, \mu \in \mathbb{Z}$. Then, using the forms

$$X = \begin{pmatrix} X_1 & 0 \\ V_1 & \overline{X_1} \end{pmatrix}, Y = \begin{pmatrix} Y_1 & 0 \\ W_1 & \overline{Y_1} \end{pmatrix}, a_{d,d} = \begin{pmatrix} a_{d-1,d-1} & 0 \\ a_{d-1,d-1} & (a_{d-1,d-1})^{-1} \end{pmatrix},$$

we expand

$$\lambda X + \mu Y a_{d,d} = \begin{pmatrix} \lambda X_1 + \mu Y_1 a_{d-1,d-1} & 0 \\ \lambda V_1 + \mu (W_1 + \overline{Y_1}) a_{d-1,d-1} & \lambda \overline{X_1} + \mu \overline{Y_1} (a_{d-1,d-1})^{-1} \end{pmatrix}.$$

Suppose

$$\lambda X + \mu Y a_{d,d} = 0;$$

we want to show $\lambda X = \mu Y = 0$. Now

$$\begin{aligned}
\lambda X_1 + \mu Y_1 a_{d-1,d-1} &= 0, \quad \lambda \overline{X_1} + \mu \overline{Y_1} (a_{d-1,d-1})^{-1} = 0, \\
\lambda V_1 + \mu (W_1 + \overline{Y_1}) a_{d-1,d-1} &= 0
\end{aligned}$$

and by induction,

$$\begin{aligned}\lambda X_1 &= \mu Y_1 = \lambda \overline{X_1} = \mu \overline{Y_1} = 0, \\ \lambda V_1 &= \mu (W_1 + \overline{Y_1}) = 0, \\ \mu W_1 &= 0.\end{aligned}$$

□

Thus

$$\begin{aligned}\mathbf{A}_d &= \mathbb{Z} \langle a_1, \dots, a_{d-1} \rangle + \mathbb{Z} \langle a_1, \dots, a_{d-1} \rangle a_d, \\ (\mathbf{A}_d)^\varphi &= (\mathbb{Z} \langle a_1, \dots, a_{d-1} \rangle)^\varphi + (\mathbb{Z} \langle a_1, \dots, a_{d-1} \rangle)^\varphi a_{d,d} \\ &= U_d \oplus U_d \alpha_{d,d}.\end{aligned}$$

By induction, the ring \mathbf{A}_{d-1} and the subring $\mathbb{Z} \langle a_1, \dots, a_{d-1} \rangle$ of \mathbf{A}_d are free \mathbb{Z} -modules of rank 2^{d-1} isomorphic to U_d . Therefore, from the above lemma, we reach \mathbf{A}_d is a free \mathbb{Z} -module of rank 2^d and conclude φ is a monomorphism.

(iii) The ring B_d is a free \mathbb{Z} -module freely generated by

$$\{u(a_{j_1}) \dots u(a_{j_t}) \mid 1 \leq j_1 < \dots < j_t \leq d\}$$

and $(B_d)^{d+1} = 0$,

$$(B_d)^d = \mathbb{Z} u(a_1) \dots u(a_d) \neq 0.$$

(iv) Since $u([a_i, a_j])^2 = 0$, $u(a_l)^2 = 0$, it follows that for all i, j, l ,

$$\begin{aligned}u([a_i, a_j], a_l) &= u([a_i, a_j]) u(a_l) - u(a_l) u([a_i, a_j]) \\ &= 2u(a_i) u(a_j) u(a_l) - 2u(a_l) u(a_i) u(a_j) \\ &= 2u(a_i) u(a_j) u(a_l) - 2u(a_i) u(a_j) u(a_l) \\ &= 0, \\ [a_i, a_j, a_l] &= e.\end{aligned}$$

Hence, $\gamma_3(G) = \{e\}$. As we noted before \mathbf{A}_d maps onto $\mathbb{Z} \frac{F}{F'}$ and therefore $\frac{G_d}{G'_d}$ is a free abelian group of rank d . We need to show that G'_d is free abelian of rank $\binom{d}{2}$. Clearly G'_d is an abelian group generated by $[a_i, a_j]$ for $1 \leq i < j \leq d$. Since $[a_i, a_j] = 1 + 2u(a_i) u(a_j)$ and the set $\{u(a_i) u(a_j) \mid 1 \leq i < j \leq d\}$ is \mathbb{Z} -linearly independent, it follows G_d is a free d -generated nilpotent group of class 2.

4. CUBIC UNIPOTENT CONDITIONS

Let R be a ring with unity and let $G = \langle a_i \mid (1 \leq i \leq d) \rangle$ be a multiplicative subgroup of R . Then for $g_1, g_2, \dots, g_n \in G$,

$$u(g_1 g_2 \dots g_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_s \leq n} u(g_{i_1}) u(g_{i_2}) \dots u(g_{i_s}).$$

Suppose $(g-1)^3 = 0$. Then, for $m \geq 1$,

$$\begin{aligned}
g^m &= (1 + u(g))^m = 1 + mu(g) + \binom{m}{2}u(g)^2, \\
u(g^m) &= g^m - 1 = mu(g) + \binom{m}{2}u(g)^2, \\
u(g^m)^2 &= m^2u(g)^2, u(g^m)^3 = 0.
\end{aligned}$$

For negative exponents, we use

$$\begin{aligned}
g^{-1} &= g^2 - 3g + 3, \\
u(g^{-1}) &= u(g^2) - 3u(g) = -u(g) + u(g)^2, \\
u(g^{-1})^2 &= u(g)^2.
\end{aligned}$$

Thus, for $m \geq 1$,

$$\begin{aligned}
u(g^{-m}) &= mu(g^{-1}) + \binom{m}{2}u(g^{-1})^2 = m(-u(g) + u(g)^2) + \binom{m}{2}u(g)^2 \\
&= -mu(g) + \binom{m+1}{2}u(g)^2.
\end{aligned}$$

Using $\binom{-i+1}{2} = \frac{(-i+1)(-i)}{2} = \frac{(i-1)i}{2} = \binom{i}{2}$, we conclude

$$u(g^m) = mu(g) + \binom{m}{2}u(g)^2$$

for all m .

4.1. The 2-generated case. We analyze directly the ring

$$\mathbf{A} = \frac{\mathbb{D}F_2}{\langle (x-1)^3 \mid x \in P \rangle}, P = \{y_1, y_2, y_1y_2, y_1^{-1}y_2, y_1^2y_2, y_1y_2^2, [y_1, y_2]\}.$$

Note that $(y_1^{-1}y_2 - 1)^3 = 0$ implies $(y_1y_2^{-1} - 1)^3 = 0$. We denote $u(a) = a - 1, u(b) = b - 1$ in \mathbf{B} by U, V .

4.1.1. Proof of Theorem 3, (i), (ii), (iii). We start with the relations $u(a)^3 = u(b)^3 = u(ab)^3 = u(a^{-1}b)^3 = u(a^2b)^3 = u(ab^2)^3 = 0$ which translate to $U^3 = V^3 = 0, (UV + U + V)^3 = 0$. The interchange $a \leftrightarrow b$ translates to $U \leftrightarrow V$, the substitution $ab \rightarrow a^{-1}b$ translates to $U \rightarrow U^2 - U, V \rightarrow V$ and similarly, $ab \rightarrow a^2b$ translates to $U \rightarrow U^2 + 2U, V \rightarrow V$.

We also order $U < V < U^2 < V^2$. Monomials are ordered decreasingly as follows: by syllable length, then by total length, then by total V length and finally, by reverse lexicographical ordering.

In the sequence of manipulations below we use the following notation: given $m, p \in \mathbf{B}$ where m is a monomial and where the monomial summands of p are of lesser order than p , a relation (or, rule of substituting m by p) is of the form $m = p$. Given a monomial $q \in \mathbf{B}$ written as a sum of monomials $q = \dots + c * m + \dots$ ($c \in \mathbb{Z}$) then $q \leftarrow m = p$ (abbreviated as $q \leftarrow m$) indicates the substitution of m by p in the expression for q . If $q = \dots + c * m' * m * m'' + \dots$ where one of the monomials m', m'' is non-empty, then $q \leftrightsquigarrow m = p$ (abbreviated as $q \leftrightsquigarrow m$) indicates the substitution of m by p in q . Composing substitutions is done from left to right and we write $q \leftarrow m_1, m_2$ for $(q \leftarrow m_1) \leftarrow m_2$.

$$(1) := (UV + U + V)^3 = 0$$

$$UVUVUV = -VUVUV - UVUVU - UVUV^2 - UV^2UV - UVU^2V - U^2VUV - 2UVUV - VUVU - VUV^2 - V^2UV - VU^2V - UV^2U - UVU^2 - U^2VU - VUV - UVU - U^2V^2 - UV^2 - V^2U - U^2V - VU^2;$$

$$(2) := (1) \times V^2 \Leftarrow (1) \Leftarrow (1)$$

$$UVUVU = -UVUV - VUVU - UV^2U - UVU^2 - U^2VU - VUV - UVU - UV^2 - V^2U - U^2V - VU^2;$$

$$(3) := (2) \times U^2 \Leftarrow (2) \Leftarrow (2)$$

$$UVUV = -VUV - UVU + V^2U^2 - UV^2 - V^2U - U^2V - VU^2;$$

$$(4) := U^2 \times (3) \Leftarrow (3) \Leftarrow (3)$$

$$U^2V^2U^2 = UV^2U^2 + U^2V^2U + U^2VU^2 - UV^2U - UVU^2 - U^2VU + VUV + UVU - U^2V^2 - V^2U^2 + UV^2 + V^2U + U^2V + VU^2;$$

$$(5) := (4) \Leftarrow (U \mapsto U^2 - U)$$

$$VU^2V = UV^2U + VUV - UVU + UV^2 + V^2U - U^2V - VU^2;$$

$$(6) := (4) \Leftarrow (V \mapsto V^2 - V)$$

$$V^2UV^2 = U^2VU^2 + VUV^2 + V^2UV - UVU^2 - U^2VU - VUV + UVU - UV^2 - V^2U + U^2V + VU^2;$$

$$(7) := (4) \Leftarrow (V \mapsto V^2 + 2V) \Leftarrow (6) \Leftarrow (4) \Leftarrow * \frac{1}{3}$$

$$VUV^2 = -V^2UV;$$

$$(8) := (7) \times V$$

$$V^2UV^2 = 0;$$

$$(9) := (7) \Leftarrow (U \leftrightarrow V)$$

$$UVU^2 = -U^2VU;$$

$$(10) := (8) \Leftarrow (U \leftrightarrow V)$$

$$U^2VU^2 = 0;$$

$$(11) := (6) \Leftarrow (10) \Leftarrow (9) \Leftarrow (8) \Leftarrow (7)$$

$$VUV = UVU - UV^2 - V^2U + U^2V + VU^2;$$

$$(12) := \textcolor{red}{(5)} \leftarrow \textcolor{red}{(11)} \\ VU^2V = UV^2U;$$

$$(13) := \textcolor{red}{(11)} \times \textcolor{red}{V} \leftarrow \textcolor{red}{(12)} \leftarrow \textcolor{red}{(7)} \leftarrow \textcolor{red}{(3)} \leftarrow \textcolor{red}{(11)} \\ UV^2U = 2UVU - U^2V^2 - V^2U^2 + 2U^2V + 2VU^2.$$

$$(14) := \textcolor{red}{(13)} \times \textcolor{red}{U} \rightleftharpoons \textcolor{red}{(13)} \leftarrow \textcolor{red}{(9)} \\ UV^2U^2 = -U^2V^2U$$

$$(15) := \textcolor{red}{U} \times \textcolor{red}{(14)} \\ U^2V^2U^2 = 0.$$

Further similar manipulations (see [4]) produce

$$(UV)^3 = (UV)^2 U^2 = U^2 (VU)^2 = (UV^2)^2 = (U^2V)^2 = (UVU)^2 = 0;$$

$$\begin{aligned} VUVU^2V &= UV^2UVU = UVU^2V^2 = U^2V^2UV = V^2UVU^2 = VU^2V^2U; \\ V^2U^2VU &= V^2UVU^2 = VUV^2U^2 = UV^2U^2V = U^2VUV^2 = UVUV^2U = VU^2VUV = -VU^2V^2U. \end{aligned}$$

With these relations, it is directly verifiable that monomials of length 7 are null.

Lemma 2. *Let $c_2 = [a, b]$ and $C_2 = u(c_2) = c_2 - 1$. Then $C_2^3 = 6UV^2U^2V = 0$.*

Proof. Since for a general element g of the group, $g^{-1} = u(g)^2 - u(g) + 1$,

$$C_2 = (U^2 - U + 1)(V^2 - V + 1)(U + 1)(V + 1) - 1.$$

On expanding the above expression and eliminating monomials of length 7 we get

$$\begin{aligned} C_2^3 &= (UV)^3 - (VU)^3 \\ &\quad + UV^2UVU - UVUV^2U \\ &\quad + VUVU^2V - VU^2VUV \\ &\quad + VU^2V^2U - UV^2U^2V. \end{aligned}$$

Then, with further easy substitutions from the relations listed above, we obtain

$$C_2^3 = 6U^2V^2UV.$$

□

4.1.2. *Table of relations for the augmentation ideal \mathbf{B} .*

$$\begin{aligned}
(4.1) \quad VUV &= UVU - UV^2 + U^2V - V^2U + VU^2, \\
(4.2) \quad (UV)^2 &= -2UVU - 2U^2V + V^2U^2 - 2VU^2, \\
(4.3) \quad (VU)^2 &= -2UVU + U^2V^2 - 2U^2V - 2VU^2, \\
(4.4) \quad (VU)^2 - (UV)^2 &= U^2V^2 - V^2U^2, \\
(4.5) \quad UV^2U &= 2(UVU + U^2V + VU^2) - U^2V^2 - V^2U^2, \\
(4.6) \quad (UV)^2 + UV^2U &= -U^2V^2, \\
(4.7) \quad VU^2V &= UV^2U, \\
(4.8) \quad UVU^2 &= -U^2VU, \\
(4.9) \quad VUV^2 &= -V^2UV, \\
(4.10) \quad U^2VU^2 &= 0, V^2UV^2 = 0, \\
(4.11) \quad UVUVU &= 0, \\
(4.12) \quad VU^2V^2U &= U^2VU^2V = 0. \\
(4.13) \quad V^2UVU &= UV^2UV = -V^2U^2V, \\
(4.14) \quad VUV^2U &= UVUV^2 = V^2U^2V, \\
(4.15) \quad VU^2VU &= U^2VUV = -U^2V^2U, \\
(4.16) \quad VUVU^2 &= UVU^2V = U^2V^2U.
\end{aligned}$$

Analysis of the interdependence of the relations in \mathbf{B} produces the following (see, [4]),

Lemma 3. *The ring \mathbf{B} is defined over \mathbb{D} by its generators U, V with relations*

$$\begin{aligned}
VUV &= UVU - UV^2 + U^2V - V^2U + VU^2, \\
UV^2U &= 2(UVU + U^2V + VU^2) - U^2V^2 - V^2U^2, \\
VU^2V &= UV^2U, \\
UVU^2 &= -U^2VU, \\
VU^2V^2U &= U^2VU^2V = 0.
\end{aligned}$$

The rank of the ring \mathbf{A} is now at most 18, and is \mathbb{D} -generated by

$$\begin{aligned}
&\{1, U, V, U^2, V^2, UV, VU, \\
&U^2V, VU^2, UV^2, V^2U, V^2U^2, U^2V^2, \\
&VUV, U^2VU, V^2UV, \\
&U^2V^2U, V^2U^2V\}.
\end{aligned}$$

On rewriting the generating set as

$$\begin{aligned}
&\{1, U, V, U^2, UV, VU, V^2, U^2V, UV^2, VU^2, V^2U, U^2VU, U^2V^2, VU^2V, V^2U^2, V^2UV, \\
&U^2V^2U, V^2U^2V\}
\end{aligned}$$

we find that a, b are represented by the following upper triangular matrices

$$\begin{aligned}
a \mapsto & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
b \mapsto & \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

With the use of GAP [5], it is shown that the matrices satisfy the relations and thus $\text{rank}_{\mathbb{D}}(\mathbf{A}) = 18$, \mathbf{B} is nilpotent of degree 6 and the group G is nilpotent of degree 4.

To prove that G is free 2-generated of nilpotency class 4, we find the expressions for the basic commutators in terms of the basis elements of \mathbf{B} [4]:

- (1) $[a, b] - 1 = UV - VU - 6U^2V + UV^2 - 5VU^2 + 2V^2U - 4UVU + U^2V^2 + 2V^2U^2 - U^2VU - VUV^2 + 2U^2V^2U - VU^2V^2,$
- (2) $[a, b, a] - 1 = -3U^2V - 3VU^2 + 3V^2U^2 - 6U^2VU + 3U^2V^2U,$
- (3) $[a, b, b] - 1 = 3UV^2 + 3V^2U - 3U^2V^2 + 6V^2UV - 3V^2U^2V,$
- (4) $[a, b, a, a] - 1 = -6U^2VU + 3U^2V^2U,$

- (5) $[a, b, b, a] - 1 = [a, b, a, b] - 1 = -3U^2V^2 + 3V^2U^2,$
- (6) $[a, b, b, b] - 1 = 6V^2UV - 3V^2U^2V.$
- (7) $[a, b, *, *, *] = 1.$

Since the set of monomials of minimal weight in the above commutators form an independent set, it follows that G is free nilpotent of class 4.

4.2. Proof of Theorem 3 (iv).

Lemma 4. *The set of solutions of $X^3 = 0$ in \mathbf{B} are described by the algebraic affine variety \mathcal{V} over \mathbb{D} , given by equation*

$$\frac{x_1x_2(x_1+x_2)}{2} - x_1x_2x_3 - x_1x_2x_4 + x_2^2x_5 + x_1^2x_6 = 0. (*)$$

Proof. Let

$$X = x_1V + x_2U + x_3VU + x_4UV + x_5V^2 + x_6U^2 + x_7V^2U + x_8VU^2 + x_9UV^2 + x_{10}U^2V + W_0,$$

where $W_0 \in \mathbf{B}^4$. Since $\mathbf{B}^6 = 0$, we deduce that $X^3 = \sum x_i x_j x_k f_{ijk}$ where $1 \leq i \leq j \leq k \leq 10$ and no two indices belong to $\{7, 8, 9, 10\}$. Furthermore, $f_{ijk} = 0$ for $i = 1, 2; j, k = 3, \dots, 6$; for example,

$$\begin{aligned} f_{134} &= V^2U^2V + VU^2V^2 + VUVVU + UV^2UV + VUV^2U = V^2U^2V + VU^2V^2 + UV^2UV + VUV^2U \text{ (by(4.7))} \\ &= V^2U^2V + 2UV^2UV + VUV^2U \text{ (by(4.7))} = 2V^2U^2V + 2UV^2UV \text{ (by(4.14))} = 0 \text{ (by(4.15)).} \end{aligned}$$

The expression for X^3 becomes

$$\begin{aligned} X^3 &= x_1^2x_2(V^2U + VUV + UV^2) + x_1x_2^2(VU^2 + UVU + U^2V) + x_1^2x_3(V^2UV + VUV^2) \\ &+ x_1x_2x_3(V^2U^2 + UVVU + 2VU^2V + 2VUVU) + x_2^2x_3(U^2VU + UVU^2) + \\ &+ x_1^2x_4(V^2UV + VUV^2) + x_1x_2x_4(U^2V^2 + VUVU + 2UV^2U + 2UVUV) + \\ &+ x_2^2x_4(U^2VU + UVU^2) + x_1^2x_6(V^2U^2 + VU^2V + U^2V^2) + x_1x_2x_5(V^2UV + VUV^2) + \\ &+ x_2^2x_5(U^2V^2 + V^2U^2 + UV^2U) + x_1x_2x_6(U^2VU + UVU^2) + \\ &+ x_1^2x_7(V^2UV^2) + x_1x_2x_7(UV^2UV + V^2U^2V + V^2UVU + VUV^2U) + \\ &+ x_2^2x_7(U^2V^2U + UV^2U^2) + x_1^2x_8(V^2U^2V + VU^2V^2) + x_1x_2x_8(VU^2VU + UV^2U^2 + VUVU^2 + UVU^2V) + \\ &+ x_2^2x_8(U^2VU^2) + x_1^2x_9(U^2V^2U + UV^2U^2) + x_1x_2x_9(UV^2UV + VU^2V^2 + UVUV^2 + VUV^2U) + \\ &+ x_2^2x_9(V^2UV^2) + x_1^2x_{10}(U^2VU^2) + x_1x_2x_{10}(VU^2VU + U^2V^2U + U^2VUV + UVU^2V) + \\ &+ x_2^2x_{10}(V^2U^2V + VU^2V^2) + x_1x_3^2(V^2VUV + VUVVU + VUV^2U) + x_2x_3^2(UVUVU + VU^2VU + VUVU^2) + \\ &+ x_1x_3x_4(V^2U^2V + VU^2V^2 + VUVUV + UV^2UV + VUV^2U) + x_2x_5^2V^2UV^2 + \\ &+ x_1x_4^2(UV^2UV + VUVUV + UVUV^2) + x_2x_4^2(U^2VUV + UVUVU + UVU^2V) + \\ &+ x_1x_3x_5V^2UV^2 + x_2x_3x_5(UVUV^2 + VU^2V^2 + V^2UVU + VUV^2U) + \\ &+ x_1x_4x_5V^2UV^2 + x_2x_4x_5(UV^2UV + UVUV^2 + V^2U^2V + V^2UVU) + \\ &+ x_1x_5x_6(V^2U^2V + VU^2V^2). \end{aligned}$$

On using our table of relations, this expression reduces to

$$X^3 = (x_1^2x_2 + x_1x_2^2 - 2x_1x_2x_4 - 2x_1x_2x_3 + 2x_1^2x_6 + 2x_2^2x_5)(UVU + U^2V + VU^2).$$

□

Lemma 5. *In the ring \mathbf{A} , we have $X^3 = 0$ for every $1 + X \in G = \langle 1 + U, 1 + V \rangle$.*

Proof. It is easy to see that $I = \mathbf{B}$ is a group with respect to the operation $a \circ b = a + b + ab$. Unfortunately, \mathcal{V} is not a subgroup of I . The following steps lead to $G \subset \mathcal{V}$.

Step 1. For any $n, m \in \mathbf{N}$

$$(1 + U)^n(1 + V)^m = 1 + S, S \in \mathcal{V}.$$

Step 2. If $1 + C \in [G, G]$, then

$$C = y(VU - UV) + C_3, C_3 \in I^3.$$

Step 3. If $X \in \mathcal{V}$, $C \in [G, G]$, then $X \circ C \in \mathcal{V}$. Indeed, let $X = x_1V + x_2U + x_3VU + x_4UV + x_5V^2 + x_6U^2 + W$, $W \in \mathbf{B}^3$. Then

$$\begin{aligned} X \circ C &= X + C + XC \\ &= x_1V + x_2U + (x_3 + y)VU + (x_4 - y)UV + x_5V^2 + x_6U^2 + W_1, \\ W_1 &\in \mathbf{B}^3 \end{aligned}$$

and the coefficients of this element satisfy the equation (*).

Finally, since \mathbf{B} is nilpotent, every element of G has the form $(1 + U)^n(1 + V)^m(1 + C)$, $1 + C \in [G, G]$ and the proof follows. \square

4.3. The d -generated case. Recall F_d is the free group of rank d generated by $\{y_1, \dots, y_d\}$. Let $x_i \in \{y_1^{\pm 1}, \dots, y_d^{\pm 1}\}$ ($1 \leq i \leq m$) and $r = x_1x_2\dots x_m$ be a reduced word. Then $\{x_1, x_1x_2, \dots, (x_1x_2\dots x_i), \dots, r\}$ is the set of initial segments of r .

The following formula holds in the augmentation ideal of $\mathbb{Z}F_d$,

$$u(r) = u(x_1\dots x_m) = \sum_{1 \leq i_1 < \dots < i_t < m} u(x_{i_1}) \dots u(x_{i_t}) + u(x_1) \dots u(x_m).$$

The monomial $\nu = u(x_1) \dots u(x_m)$ in the above expression is the leading term and it is of highest weight. On the other hand, given a monomial ν , there exists $r \in F_d$, as above, for which ν is a leading monomial of $u(r)$; write $r = r_\nu(y_i)$.

As we assume that $u(a_i)^3 = 0$ for all i , with the notation $u(a_i) = U_i$, our monomials in the quotient ring are of the form $\nu = U_{i_1}^{\varepsilon_1} U_{i_2}^{\varepsilon_2} \dots U_{i_s}^{\varepsilon_s}$ where $U_{i_t} \neq U_{i_{t+1}}$ and $\varepsilon_i = 1, 2$. Furthermore, since $u(a_i^{-1}) = -U_i + U_i^2$, we can choose $r_\nu = r_\nu(a_i)$ to be $a_{i_1}^{\delta_1} a_{i_2}^{\delta_2} \dots a_{i_s}^{\delta_s}$ where $a_{i_t} \neq a_{i_{t+1}}$, $\delta_i = 1$ for $\varepsilon_i = 1$ and $\delta_i = -1$ for $\varepsilon_i = 2$.

We denote the augmentation $\omega(\mathbf{A}_d)$ by ω_d .

4.3.1. Proof of Theorem 4. For $d = 1$, we take $\mathbf{A}_1 = \frac{\mathbb{D}F_1}{\langle (x-1)^3 | x \in P_1 \rangle}$ where $P_1 = \{y_1\}$; the augmentation ideal ω_1 is the \mathbb{D} -span of $\mathbf{W}_1 = \{U_1, U_1^2\}$. For $d = 2$, we take $\mathbf{A}_2 = \mathbf{A} = \frac{\mathbb{D}F_2}{\langle (x-1)^3 | x \in P \rangle}$ where $P = \{y_1, y_2, y_1y_2, y_1^{-1}y_2, y_1^2y_2, y_1y_2^2\}$. We recall that the following relations which hold in $\omega_2 = \omega(\mathbf{A}_2)$,

$$\begin{aligned} U_2U_1U_2^2 &= -U_2^2U_1U_2 \\ U_2U_1U_2 &= U_1U_2U_1 - U_1U_2^2 + U_1^2U_2 - U_2^2U_1 + U_2U_1^2. \end{aligned}$$

We conclude

$$U_2\mu U_2^2 = -U_2^2\mu U_2, U_2^2\mu U_2^2 = 0$$

and $U_2\mu U_2$ is a sum of monomials $\mu_1 U_2 \mu_2, \mu'_1 U_2^2, U_2^2 \mu'_2$ where $\mu_1, \mu_2, \mu'_1, \mu'_2$ are monomials in ω_1 . Therefore ω_2 is \mathbb{D} -generated by the monomials

$$\begin{aligned} \mathbf{W}_2 &= \mathbf{W}_1 \cup \{U_2^\varepsilon\} \cup \mathbf{W}_1 U_2^\varepsilon \cup U_2^\varepsilon \mathbf{W}_1 \\ &\quad \cup U_2^2 \mathbf{W}_1 U_2 \cup U_2^2 \mathbf{W}_1 U_2 \mathbf{W}_1 \\ &\quad \cup \mathbf{W}_1 U_2^2 \mathbf{W}_1 U_2 \\ &\quad \cup \mathbf{W}_1 U_2^2 \mathbf{W}_1 U_2 \mathbf{W}_1 \end{aligned}$$

where $\varepsilon = 1, 2$.

We note that \mathbf{W}_2 is somewhat larger than we had produced earlier. However, \mathbf{W}_2 lends itself to easier generalization.

Suppose \mathbf{A}_d is defined such that ω_d is generated by a finite set of non-zero monomials $\mathbf{W}_d = \{\nu_1(a_i), \dots, \nu_t(a_i)\}$. Define \widetilde{P}_d to be the closure of

$$P_d \cup \{r_{\nu_1}(y_i), \dots, r_{\nu_t}(y_i)\}$$

under taking initial segments in F_d . Then

$$\begin{aligned} P_{d+1} &= \widetilde{P}_d \\ &\quad \cup \left\{ e, h^{\pm 1} \mid h \in \widetilde{P}_d \right\} y_{d+1} \\ &\quad \cup \widetilde{P}_d y_{d+1}^2 \\ &\quad \cup \left\{ h^2 \mid h \in \widetilde{P}_d \right\} y_{d+1} \end{aligned}$$

and we define

$$\mathbf{A}_{d+1} = \frac{\mathbb{D}F_{d+1}}{\left\langle (x-1)^3 \mid x \in P_{d+1} \right\rangle}.$$

We will write $b = a_{d+1}$ in the ring $\mathbf{A}_{d+1} = \frac{\mathbb{D}F_d}{\left\langle (x-1)^3 \mid x \in P_{d+1} \right\rangle}$ and write $V = U_{d+1}$.

Thus, for every $q \in \widetilde{P}_d$ the subring $\langle q, b \rangle$ of \mathbf{A}_{d+1} is a homomorphic image of \mathbf{A}_2 via $a \rightarrow q, b \rightarrow b$. Using the epimorphism produces

$$Vu(q)V^2 = -V^2u(q)V$$

which generalizes to

$$V\mu V^2 = -V^2\mu V, V^2\mu V^2 = 0$$

for all μ monomial in the subring $\langle U_1, \dots, U_d \rangle$. Furthermore,

$$Vu(q)V = u(q)Vu(q) - u(q)V^2 + u(q)^2V - V^2u(q) + Vu(q)^2.$$

Let $q = q_1x$ where q_1 is an initial segment of q . Then $u(q) = u(q_1) + u(x) + u(q_1)u(x)$ and, as the above formula holds for $u(q_1), u(x)$, we expand our expression to obtain a formula for $Vu(q_1)u(x_1)V$.

To illustrate, we apply the substitution

$$a_1 \rightarrow a_1a_2, b \rightarrow b$$

which translates to the substitution

$$\begin{aligned} U_1 &\rightarrow U_1 + U_2 + U_1U_2, \\ U_1^2 &\rightarrow U_2^2U_1^2 - U_2^2U_1 + U_2^2 - U_2U_1^2 + U_1U_2 + U_2U_1 + U_1^2 \\ V &\rightarrow V. \end{aligned}$$

in ω_2 .

The equation

$$VUV = UVU - UV^2 + U^2V - V^2U + VU^2$$

transforms to

$$\begin{aligned} & V(U_1 + U_2 + U_1U_2)V \\ = & (U_1 + U_2 + U_1U_2)V(U_1 + U_2 + U_1U_2) - (U_1 + U_2 + U_1U_2)V^2 \\ & + (U_2^2U_1^2 - U_2^2U_1 + U_2^2 - U_2U_1^2 + U_1U_2 + U_2U_1 + U_1^2)V \\ & - V^2(UVU - UV^2 + U^2V - V^2U + VU^2) \\ & + V(U_2^2U_1^2 - U_2^2U_1 + U_2^2 - U_2U_1^2 + U_1U_2 + U_2U_1 + U_1^2) \end{aligned}$$

which, in expanded form, is

$$\begin{aligned} & VU_1V + VU_2V + VU_1U_2V \\ = & U_1VU_1 + U_2VU_1 + U_1U_2VU_1 \\ & + U_1VU_2 + U_2VU_2 + U_1U_2VU_2 \\ & + U_1VU_1U_2 + U_2VU_1U_2 + U_1U_2VU_1U_2 \\ & - U_1V^2 - U_2V^2 - U_1U_2V^2 \\ & + U_2^2U_1^2V - U_2^2U_1V - U_2U_1^2V + U_1U_2V + U_2U_1V + U_2^2V + U_1^2V \\ & - V^2U_1 - V^2U_2 - V^2U_1U_2 \\ & + VU_2^2U_1^2 - VU_2^2U_1 - VU_2U_1^2 + VU_1U_2 + VU_2U_1 + VU_2^2 + VU_1^2. \end{aligned}$$

We separate monomials involving just U_1 or U_2 in the above:

$$\begin{aligned} VU_1U_2V = & (-VU_1V + U_1VU_1 - U_1V^2 - V^2U_1 + VU_1^2 + U_1^2V) \\ & + (-VU_2V + U_2VU_2 - U_2V^2 - V^2U_2 + VU_2^2 + U_2^2V) \\ & + U_2VU_1 + U_1U_2VU_1 \\ & + U_1VU_2 + U_1U_2VU_2 \\ & + U_1VU_1U_2 + U_2VU_1U_2 + U_1U_2VU_1U_2 \\ & - U_1U_2V^2 \\ & + U_2^2U_1^2V - U_2^2U_1V - U_2U_1^2V + U_1U_2V + U_2U_1V \\ & - V^2U_1U_2 \\ & + VU_2^2U_1^2 - VU_2^2U_1 - VU_2U_1^2 + VU_1U_2 + VU_2U_1. \end{aligned}$$

We separate monomials involving just U_1 or U_2 in the above:

$$\begin{aligned} VU_1U_2V = & (-VU_1V + U_1VU_1 - U_1V^2 - V^2U_1 + VU_1^2 + U_1^2V) \\ & + (-VU_2V + U_2VU_2 - U_2V^2 - V^2U_2 + VU_2^2 + U_2^2V) \\ & + U_2VU_1 + U_1U_2VU_1 \\ & + U_1VU_2 + U_1U_2VU_2 \\ & + U_1VU_1U_2 + U_2VU_1U_2 + U_1U_2VU_1U_2 \\ & - U_1U_2V^2 \\ & + U_2^2U_1^2V - U_2^2U_1V - U_2U_1^2V + U_1U_2V + U_2U_1V \\ & - V^2U_1U_2 \\ & + VU_2^2U_1^2 - VU_2^2U_1 - VU_2U_1^2 + VU_1U_2 + VU_2U_1. \end{aligned}$$

Therefore,

$$\begin{aligned}
VU_1U_2V = & \\
& U_1U_2VU_1U_2 \\
& + U_1U_2VU_1 + U_1U_2VU_2 \\
& + U_1VU_1U_2 + U_2VU_1U_2 \\
& + U_1U_2V + U_1VU_2 + VU_1U_2 \\
& + U_2U_1V + U_2VU_1 + VU_2U_1 \\
& + U_2^2U_1V + VU_2^2U_1 \\
& - U_2^2U_1V - VU_2^2U_1 - U_2U_1^2V - VU_2U_1^2 \\
& - U_1U_2V^2 - V^2U_1U_2.
\end{aligned}$$

The summands are of the form $\mu_1V\mu_2, \mu_1'V^2, V^2\mu_2'$ where $\mu_1, \mu_2, \mu_1', \mu_2'$ are monomials in the subring $\mathbf{C}_2 = \langle U_1, U_2 \rangle$ and each of $l(\mu_1) + l(\mu_2), l(\mu_1'), l(\mu_2')$ is at least $l(U_1U_2) = 2$.

Thus, we derive the fact: for every μ monomial in the subring $\mathbf{C}_d = \langle U_1, \dots, U_d \rangle$, of length $l(\mu) = m$, the monomial $V\mu V$ decomposes as the sum of monomials of the form $\mu_1V\mu_2, \mu_1'V^2, V^2\mu_2'$ for some monomials $\mu_1, \mu_2, \mu_1', \mu_2'$ in the subring $\langle U_1, \dots, U_d \rangle$ such that each of $l(\mu_1) + l(\mu_2), l(\mu_1'), l(\mu_2')$ is at least m .

Given the above, we conclude that

$$\begin{aligned}
\mathbf{W}_{d+1} = & \mathbf{W}_d \cup \{U_{d+1}^\varepsilon\} \cup \mathbf{W}_d U_{d+1}^\varepsilon \cup U_{d+1}^\varepsilon \mathbf{W}_d \\
& \cup U_{d+1}^2 \mathbf{W}_d U_{d+1} \cup U_{d+1}^2 \mathbf{W}_d U_{d+1} \mathbf{W}_d \\
& \cup \mathbf{W}_d U_{d+1}^2 \mathbf{W}_d U_{d+1} \\
& \cup \mathbf{W}_d U_{d+1}^2 \mathbf{W}_d U_{d+1} \mathbf{W}_d
\end{aligned}$$

where $\varepsilon = 1, 2$.

A monomial in ω_{d+1} has the form $W = \mu_1 V^{i_1} \mu_2 V^{i_2} \dots \mu_k V^{i_k}$ where each μ_i is a monomial in the subring \mathbf{C}_d . Define $L(W) = \sum_{1 \leq j \leq k} l(\mu_j)$. Therefore W is a sum of monomials of the form $\sigma V^{2s} \sigma' V^t \sigma''$ where $\sigma, \sigma', \sigma''$ are monomials in \mathbf{C}_d , with $s, t \in \{0, 1\}$ and $l(\sigma) + l(\sigma') + l(\sigma'') \geq L(W)$.

Suppose the degree of nilpotency of ω_d is ρ_d and $L(W) \geq 3\rho_d$. Then one of $l(\sigma), l(\sigma'), l(\sigma'')$ is at least ρ_d and therefore one of $\sigma, \sigma', \sigma''$ is zero.

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